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Concircular Curvature Tensor and Fluid Spacetimes

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Abstract In the differential geometry of certain F-structures, the importance of concircular curvature tensor is very well known. The relativistic significance of this tensor has been explored here. The spacetimes satisfying Einstein field equations and with vanishing concircular curvature tensor are considered and the existence of Killing and conformal Killing vectors have been established for such spacetimes. Perfect fluid spacetimes with vanishing concircular curvature tensor have also been considered. The divergence of concircular curvature tensor is studied in detail and it is seen, among other results, that if the divergence of the concircular tensor is zero and the Ricci tensor is of Codazzi type then the resulting spacetime is of constant curvature. For a perfect fluid spacetime to possess divergence-free concircular curvature tensor, a necessary and sufficient condition has been obtained in terms of Friedmann-Robertson-Walker model.

Keywords Concircular curvature tensor · Codazzi equation · Perfect fluid spacetime · FRW-model

1 Introduction

In general, a geodesic circle (a curve whose first curvature is constant and second curvature is identically zero) does not transform into a geodesic circle by the conformal transformation

$$\tilde{g}_{ij} = \psi^2 g_{ij} \tag{1.1}$$

of the fundamental tensor g_{ij}. The transformation which preserves geodesic circles was first introduced by Yano [9]. The conformal transformation (1.1) satisfying the partial differential

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equation

$$\psi_{;i;j} = \phi g_{ij} \tag{1.2}$$

changes a geodesic circle into a geodesic circle. Such a transformation is known as the concircular transformation and the geometry which deals with such transformation is called the concircular geometry (cf., [9]).

A (1, 3) type tensor $M(X, Y, Z)$ which remain invariant under concircular transformation, for a n-dimensional Riemannian space V_n, is given by Yano and Kon [11]

$$M(X, Y, Z) = R(X, Y, Z) - \frac{R}{n(n-1)}[Xg(Y, Z) - Yg(X, Z)] \tag{1.3}$$

where $R(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z$ (D being the Riemannian connection) is the Riemann curvature tensor and R, the scalar curvature. In local coordinates, (1.3) can be expressed as

$$M^t_{jih} = R^t_{jih} - \frac{R}{n(n-1)}(\delta^t_h g_{ji} - \delta^t_i g_{jh}) \tag{1.4}$$

Also

$$M(X, Y, Z, W) = R(X, Y, Z, W) - \frac{R}{n(n-1)}[g(X, W)g(Y, Z) - g(Y, W)g(X, Z)] \tag{1.5}$$

or, in local coordinates

$$M_{kjih} = R_{kjih} - \frac{R}{n(n-1)}[g_{kh}g_{ji} - g_{jh}g_{ki}] \tag{1.6}$$

The curvature tensor $M(X, Y, Z)$ or $M(X, Y, Z, W)$ defined through (1.3) or (1.5) is known as concircular curvature tensor. The contraction of (1.4) over t and h leads to

$$M_{ji} = R_{ji} - \frac{R}{n}g_{ji} \tag{1.7}$$

which is also invariant under concircular transformation. Moreover,

$$g^{ji}M_{ji} = 0$$

A manifold whose concircular curvature vanishes at every point is called a concircularly flat manifold.

The importance of concircular transformation and concircular curvature tensor is very well known in differential geometry of certain F-structures such as complex, almost complex, Kähler, almost Kähler, contact and almost contact structures etc. (cf., [2, 8, 11]). In this paper, the relativistic significance of this tensor has been investigated. The spacetimes satisfying Einstein field equations (with and without a cosmological term) and having zero concircular curvature tensor have been studied and the existence of Killing and conformal Killing vectors have been established for such spacetimes. The divergence of concircular curvature tensor has also been studied in detail for spacetimes with and without a perfect fluid source. It is seen (among many other results) that if concircular curvature is divergence-free and the Ricci tensor is of Codazzi type then the spacetime is of constant curvature; while for a perfect fluid spacetime having divergence-free concircular curvature tensor, a necessary and sufficient condition has been obtained in terms of Friedmann-Robertson-Walker model.

2 Spacetimes with Vanishing Concircular Curvature Tensor

Let V_4 be the spacetime of general relativity, then from (1.4) and (1.6), we have

$$M^t_{jih} = R^t_{jih} - \frac{R}{12}(\delta^t_h g_{ji} - \delta^t_i g_{jh}) \tag{2.1}$$

$$M_{kjih} = R_{kjih} - \frac{R}{12}(g_{kh} g_{ji} - g_{jh} g_{ki}) \tag{2.2}$$

From these equations, it is evident that a V_4 with vanishing concircular curvature tensor is a space of constant curvature. Thus, the concircular curvature tensor can be thought of as the failure of a Riemannian manifold V_4 to be of constant curvature. The deviation from the concircular flatness is measured by (cf., [10])

$$M = \sup_A \frac{|M_{kjih} A^{kj} A^{ih}|}{A_{ij} A^{ij}}$$

where $A^{kj} = Y^k X^j - Y^j X^h$, X, Y being two mutually orthogonal unit vectors. Also

$$A_{ij} = -A_{ji}$$

Also, if the concircular curvature tensor vanishes identically then from (2.1) for a V_4, we have

$$R_{jh} = \frac{R}{4} g_{jh} \tag{2.3}$$

which shows a concircularly-flat spacetime is an Einstein space. This equation in index-free notation, can be expressed as

$$\mathrm{Ric}(X, Y) = \frac{R}{4} g(X, Y) \tag{2.3a}$$

Remark The spaces of constant curvature play a significant role in cosmology. The simplest cosmological model is obtained by making the assumption that the universe is isotropic and homogeneous. This is known as cosmological principle. This principle, when translated into the language of Riemannian geometry, asserts that the three dimensional position space is a space of maximal symmetry [7], that is, a space of constant curvature whose curvature depends upon time. The cosmological solutions of Einstein equations which contain a three dimensional space-like surface of a constant curvature are the Robertson-Walker metrics, while four dimensional space of constant curvature is the de Sitter model of the universe (cf., [6, 7]).

In general theory of relativity, the curvature tensor describing the gravitational field consists of two parts, viz., the matter part and the free gravitational part. The interaction between these parts is described through Bianchi identities. For a given distribution of matter, the construction of gravitational potential satisfying Einstein field equations is the principal aim of all investigations in gravitational physics; and this has often been achieved by imposing symmetries on the geometry compatible with the dynamics of the chosen distribution of matter. The geometrical symmetries of the spacetime are expressed through the equation

$$\mathcal{L}_\xi A - 2\Omega A = 0 \tag{2.4}$$

where A represents a geometrical/physical quantity, $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the vector field ξ and Ω is a scalar.

One of the most simple and widely used example is the metric inheritance symmetry for which $A = g_{ij}$ in (2.4); and for this case ξ^a is Killing vector (KV) if Ω is zero (for a comprehensive review of symmetry inheritance and related results, see [1]). We shall now investigate the role of such symmetry inheritance for a space V_4 with vanishing concircular curvature tensor.

Let $\mathrm{Ric}(X, Y) \neq 0$ and the Einstein field equations with a cosmological term are

$$\mathrm{Ric}(X, Y) - \frac{1}{2} R g(X, Y) + \Lambda g(X, Y) = kT(X, Y) \tag{2.5}$$

for all vector fields X, Y. Here $\mathrm{Ric}(X, Y)$ denotes the Ricci tensor, $T(X, Y)$ is the energy-momentum tensor, Λ the cosmological constant and k the non-zero gravitational constant. From (2.3a), (2.5) leads to

$$\left(\Lambda - \frac{1}{4} R\right) g(X, Y) = kT(X, Y) \tag{2.6}$$

Since R is constant for a concircularly-flat spacetime, taking the Lie derivative of both sides of (2.6) we get

$$\left(\Lambda - \frac{1}{4} R\right) (\mathcal{L}_\xi g)(X, Y) = k(\mathcal{L}_\xi T)(X, Y) \tag{2.7}$$

If ξ is a Killing vector, then

$$(\mathcal{L}_\xi g)(X, Y) = 0 \tag{2.8}$$

and thus (2.7) leads to

$$(\mathcal{L}_\xi T)(X, Y) = 0 \tag{2.9}$$

Conversely, if (2.9) holds and since Λ, k and R are constants then from (2.7) we get ξ as a Killing vector. We can thus state the following:

Theorem 1 *For a concircularly-flat spacetime satisfying Einstein field equations with a cosmological term, there exists a Killing vector field ξ if and only if the Lie derivative of the energy-momentum tensor with respect to ξ vanishes.*

Since concircular curvature tensor is invariant under special type of conformal transformation, we shall now find the role of conformal Killing vector fields for a concircularly-flat spacetime. A vector field ξ obeying equation

$$(\mathcal{L}_\xi g)(X, Y) = 2\Omega g(X, Y) \tag{2.10}$$

is called a conformal Killing vector field, where Ω is a scalar function.

From (2.7) and (2.10), we have

$$2\left(\Lambda - \frac{1}{4} R\right) \Omega g(X, Y) = k(\mathcal{L}_\xi T)(X, Y) \tag{2.11}$$

which from (2.6) leads to

$$(\mathcal{L}_\xi T)(X, Y) = 2\Omega T(X, Y) \tag{2.12}$$

and we say that the energy-momentum tensor has the symmetry inheritance property. Conversely, if (2.12) holds then (2.10) also holds. Thus, we can state the following:

Theorem 2 *A concircularly-flat spacetime obeying the Einstein field equations with a cosmological term admits a conformal Killing vector field if and only if the energy-momentum tensor has the symmetry inheritance property.*

We shall now consider a perfect fluid spacetime with vanishing concircular curvature tensor. The energy-momentum tensor $T(X,Y)$ of a perfect fluid is given by

$$T(X,Y) = (\mu + p)A(X)A(Y) + pg(X,Y) \tag{2.13}$$

where μ is the energy density, p, the isotropic pressure and $A(X)$ is a non-zero 1-form such that $g(X,U) = A(X)$ for all X, U being the velocity vector field of the flow, that is, $g(U,U) = -1$. Also $\mu + p \neq 0$.

From (2.6) and (2.13), we have

$$\left(\Lambda - \frac{R}{4} - kp\right) g(X,Y) = k(\mu + p)A(X)A(Y) \tag{2.14}$$

A contraction of (2.14) over X and Y leads to

$$R = 4\Lambda - 3kp + k\mu \tag{2.15}$$

Now, put $X = Y = U$ in (2.14) to get

$$R = 4(\Lambda + k\mu) \tag{2.16}$$

A comparison of (2.15) and (2.16) yields $\mu + p = 0$ which means that either $\mu = 0$, $p = 0$ (empty spacetime) or the perfect fluid satisfies the vacuum-like equation of state [5]. We can thus have the following:

Theorem 3 *If a spacetime satisfying Einstein field equations with a cosmological term has vanishing concircular curvature then the matter contents of the spacetime satisfies the vacuumlike equation of state.*

The Einstein field equations are given by

$$R_{ij} - \frac{1}{2} R g_{ij} = kT_{ij} \tag{2.17}$$

Contraction of (2.17) with g^{ij} leads to

$$R = -kT \tag{2.18}$$

and (2.17) may then be expressed as

$$R_{ij} = k\left(T_{ij} - \frac{1}{2} g_{ij} T\right) \tag{2.19}$$

We know that the energy-momentum tensor for an electromagnetic field is given by

$$T_{ij} = -F_{ik}F^k_j + \frac{1}{4} g_{ij} F_{pk} F^{pk} \tag{2.20}$$

where F_{ik} represents the antisymmetric electromagnetic field tensor satisfying Maxwell's equations. From (2.20) it is evident that $T_i^i = T = 0$ and the Einstein equations for a purely electromagnetic distribution takes the form

$$R_{ij} = kT_{ij} \tag{2.21}$$

Also since $T = 0$, (2.18) yields

$$R = 0 \tag{2.22}$$

Thus, from (2.1), we have

$$R^t_{jih} = 0$$

which shows that the spacetime is flat. Hence, we have

Theorem 4 *A concircularly-flat spacetime with energy-momentum tensor of electromagnetic field is a Euclidean spacetime.*

Note This theorem points towards the conditions under which a Riemannian space can be reduced to a Euclidean space [see also Yano [9]].

3 Divergence-Free Concircular Curvature Tensor and Fluid Spacetimes

A symmetric $(0, 2)$ type tensor field T on a Riemannian manifold (M, g) is said to be a Codazzi tensor if it satisfies the Codazzi equation

$$(\nabla_X T)(Y, Z) = (\nabla_Y T)(X, Z) \tag{3.1}$$

for arbitrary vector fields X, Y and Z.

In local coordinates, this equation can be expressed as

$$T_{ij;k} = T_{ik;j} \tag{3.1a}$$

The geometrical and topological consequences of the existence of a non-trivial Codazzi tensor on a Riemannian manifold have been studied by Derdzinski and Shen [3]. The simplest Codazzi tensors are parallel one.

Taking the covariant derivative of both sides of Einstein's equation (2.17), we have

$$R_{ij;k} - R_{ik;j} = k(T_{ij;k} - T_{ik;j}) + \frac{1}{2}(g_{ij}R_{,k} - g_{ik}R_{,j}) \tag{3.2}$$

If T_{ij} is a Codazzi tensor, then (3.2) leads to

$$R_{ij;k} - R_{ik;j} = \frac{1}{2}(g_{ij}R_{,k} - g_{ik}R_{,j}) \tag{3.3}$$

which on multiplying by g^{ij} leads to

$$R^j_{k;j} = 0 \tag{3.4}$$

Thus, we have

Theorem 5 *If a spacetime satisfying Einstein field equations has an energy-momentum tensor of Codazzi type then the scalar curvature is constant.*

The divergence of concircular curvature tensor, from (2.1), can be expressed as

$$M^t_{jih;t} = R^t_{jih;t} - \frac{1}{12}(g_{ji}R_{,h} - g_{jh}R_{,i}) \tag{3.5}$$

which from contracted Bianchi identities takes the form

$$M^t_{jih;t} = R_{ji;h} - R_{jh;i} - \frac{1}{12}(g_{ji}R_{,h} - g_{jh}R_{,i}) \tag{3.6}$$

From (3.1a) and (3.6), we thus have

Theorem 6 *If the Ricci tensor is of Codazzi type and the scalar curvature is constant then the concircular curvature tensor is divergence-free.*

Theorem 7 *If the divergence of the concircular curvature tensor vanishes and the Ricci tensor is of Codazzi type, then the spacetime is of constant curvature.*

Now using (2.19) in (3.6), we have

$$M^t_{jih;t} = (T_{ji;h} - T_{jh;i}) + \frac{5k}{12}(g_{jh}T_{;i} - g_{ji}T_{;h}) \tag{3.7}$$

which lead to the following:

Theorem 8 *If the energy-momentum tensor T_{ij} is of Codazzi type and T is constant, then the spacetime has divergence-free concircular curvature tensor.*

The energy-momentum tensor of a perfect fluid is given by [cf., (2.13)]

$$T_{ji} = (\mu + p)u_j u_i + p g_{ji} \tag{3.8}$$

which leads to

$$T = -\mu + 3p \tag{3.9}$$

If $M^t_{jih;t} = 0$ and T_{ji} is a Codazzi type, then (3.7) together with (3.9) leads to

$$g_{jh}(-\mu + 3p)_{;i} - g_{ji}(-\mu + 3p)_{;h} = 0$$

which on multiplication with g^{jh} yields

$$(\mu - 3p)_{;i} = 0 \tag{3.10}$$

That is

$$(\mu - 3p) = \text{constant} \tag{3.11}$$

We thus have the following:

Theorem 9 *If for a perfect fluid spacetime, the divergence of concircular curvature tensor vanishes and the energy-momentum tensor is of Codazzi type then* $(\mu - 3p)$ *is constant.*

If the constant in (3.11) is chosen to be zero, then we have

Corollary 1 *If the energy-momentum tensor for a divergence-free concircular fluid spacetime is of Codazzi type, then the resulting spacetime is filled with radiation.*

We now consider a perfect fluid spacetime in which the divergence of the concircular curvature tensor vanishes. If $M^t_{jih;t} = 0$, then from (3.7), we have (take $k = 1$)

$$T_{ji;h} - \frac{5}{12} g_{ji} T_{;h} = T_{jh;i} - \frac{5}{12} g_{jh} T_{;i} \tag{3.12}$$

which from (3.8) and (3.9) leads to

$$\begin{aligned}(\mu + p)_{;h} u_j u_i &+ (\mu + p) u_{j;h} u_i + (\mu + p) u_j u_{i;h} + p_{;h} g_{ji} - \frac{5}{12} g_{ji} (-\mu + 3p)_{;h} \\ &= (\mu + p)_{;i} u_j u_h + (\mu + p) u_{j;i} u_h + (\mu + p) u_j u_{h;i} \\ &\quad + p_{;i} g_{jh} - \frac{5}{12} g_{jh} (-\mu + 3p)_{;i}\end{aligned} \tag{3.13}$$

Multiplication of (3.13) with u^h leads to

$$\begin{aligned}(\mu + \dot{p}) u_j u_i &+ (\mu + p)\dot{u}_j u_i + (\mu + p) u_j \dot{u}_i + \dot{p} g_{ji} - \frac{5}{12}(-\mu + 3\dot{p}) g_{ji} \\ &+ (\mu + p)_{;i} u_j + (\mu + p) u_{j;i} - p_{;i} u_j + \frac{5}{12}(-\mu + 3p)_{;i} u_j = 0\end{aligned} \tag{3.14}$$

where an overhead dot denotes covariant derivative along the fluid flow vector u_i (that is, $u^k u_{i;k} = \dot{u}_i$, etc.).

We know that the conservation equation $T^{ij}_{;j} = 0$ leads to

$$(\mu + p)\dot{u}_i = -p_{;i} - \dot{p} u_i \quad \text{(force equation)} \tag{3.15}$$

and

$$\dot{\mu} = -(\mu + p) u^i_{;i} = -(\mu + p)\theta \quad \text{(energy equation)} \tag{3.16}$$

where θ is the expansion scalar. Also, the covariant derivative of the unit flow vector can be decomposed as [4]

$$u_{i;j} = \frac{1}{3}\theta(g_{ij} + u_i u_j) - \dot{u}_i u_j + \sigma_{ij} + \omega_{ij} \tag{3.17}$$

where $\dot{u}_i$ is the acceleration vector, σ_{ij}, the symmetric shear tensor and ω_{ij}, the vorticity or rotation tensor. Using (3.15) in (3.14), we get

$$\begin{aligned}(\mu - \dot{p}) u_j u_i &+ (\mu - p)_{;i} u_j - p_{;j} u_i + \dot{p} g_{ji} + (\mu + p) u_{j;i} \\ &- \frac{5}{12}(-\mu + 3\dot{p}) g_{ji} + \frac{5}{12}(-\mu + 3p)_{;i} u_j = 0\end{aligned} \tag{3.18}$$

which on multiplication with u^j reduces to

$$(\mu - p)\dot{} u_i + (\mu - p)_{;i} + \frac{5}{12}(-\mu + 3p)\dot{} u_i + \frac{5}{12}(-\mu + 3p)_{;i} = 0 \tag{3.19}$$

This equation is satisfied only when

$$(\mu - p)_{;i} = -(\mu - p)\dot{} u_i \tag{3.20}$$

and

$$(-\mu + 3p)_{;i} = -(-\mu + 3p)\dot{} u_i \tag{3.20a}$$

Using force equations (3.15) and (3.20), we get

$$(\mu + p)\dot{u}_i = -\dot{\mu} u_i - \mu_{;i} \tag{3.21}$$

From force equations (3.15) and (3.20a), we have

$$\mu_{;i} = -3(\mu + p)\dot{u}_i - (\mu - 6p)\dot{} u_i \tag{3.22}$$

Using (3.20) in (3.18), we get

$$-p_{;j} u_i + \dot{p} g_{ji} + (\mu + p) u_{j;i} - \frac{5}{12}(-\mu + 3p)\dot{} g_{ji} + \frac{5}{12}(-\mu + 3p)_{;i} u_j = 0 \tag{3.23}$$

Contracting this equation by g^{ji}, we get

$$\dot{p} = -\frac{1}{3}(\mu + p)\theta + \frac{5}{3}(-\mu + 3p)\dot{} - \frac{5}{12}(-\mu + 3p)_{;i} u^i \tag{3.24}$$

which on using energy equation (3.16) reduces to

$$\dot{p} = \frac{1}{3}\dot{\mu} + \frac{5}{48}(-\mu + 3p)_{;i} u^i \tag{3.25}$$

Also, from force equations (3.15) and (3.23), we have

$$(\mu + p)\dot{u}_j u_i + \dot{p}(u_j u_i + g_{ji}) + (\mu + p) u_{j;i} - \frac{5}{12}(-\mu + 3p)\dot{} g_{ji} + \frac{5}{12}(-\mu + 3p)_{;i} u_j = 0 \tag{3.26}$$

Moreover, from (3.24) and (3.20a), (3.26) can be expressed as

$$(\mu + p)\left[u_{j;i} + \dot{u}_j u_i - \frac{\theta}{3}(g_{ji} + u_j u_i)\right] = 0 \tag{3.27}$$

Equation (3.27) suggests that either

$$\mu + p = 0 \tag{3.27a}$$

or

$$u_{j;i} + \dot{u}_j u_i - \frac{\theta}{3}(g_{ji} + u_j u_i) = 0 \tag{3.27b}$$

From (3.24) and (3.16), we also have

$$-\mu + 3p = \text{constant} \tag{3.27c}$$

From (3.27a) either $\mu = 0$, $p = 0$ (neither matter nor radiation) or the perfect fluid satisfies vacuumlike equation of state [5].

While from (3.27b), we have

$$u_{j;i} = \frac{\theta}{3}(u_j u_i + g_{ji}) - \dot{u}_j u_i \tag{3.28}$$

Comparing (3.17) and (3.28), we get

$$\sigma_{ij} + \omega_{ij} = 0 \tag{3.29}$$

which is satisfied only when $\sigma_{ij} = 0$, $\omega_{ij} = 0$.

Consider now (3.13) and use (3.21) and (3.27b), we have

$$p_{;h}\left(u_j u_i - \frac{1}{4}g_{ji}\right) - p_{;i}\left(u_j u_h - \frac{1}{4}g_{jh}\right) + \frac{\theta}{3}(\mu + p)(g_{jh}u_i - g_{ji}u_h) - \frac{5}{12}(\mu + p)\dot{u}_h g_{ji}$$
$$+ \frac{5}{12}(\mu + p)\dot{u}_i g_{jh} - \frac{5}{12}\dot{\mu}(g_{ji}u_h - g_{jh}u_i) = 0 \tag{3.30}$$

which on multiplication with g^{ij} reduces to

$$-\frac{7}{4}p_{;h} - \dot{p}u_h - \theta(\mu + p)u_h - \frac{5}{4}(\mu + p)\dot{u}_h - \frac{5}{4}\dot{\mu}u_h = 0 \tag{3.31}$$

Using (3.15) and (3.24), (3.31) leads to

$$(\mu + p)\dot{u}_h = 0$$

Since $\mu + p \neq 0$, thus

$$\dot{u}_h = 0 \tag{3.32}$$

Therefore, if $\mu + p \neq 0$ then the above considerations [cf., (3.27), (3.29) and (3.32)] show that the fluid is shear-free, rotation-free, acceleration-free and the energy density and pressure are constant over the spacelike hyper-surface orthogonal to the fluid four velocity. On the other hand, if the perfect fluid does not obey the vacuum-like equation of state and $(\mu + p) = 0$ (that is, $\mu = 0$, $p = 0$) then $M^t_{jih;t} = 0$. Also, from (3.8) and (3.9), we have

$$(T_{ji;h} - T_{jh;i}) + \frac{5}{12}(g_{jh}T_{;i} - g_{ji}T_{;h}) = [(\mu + p)u_j u_i + pg_{ji}]_{;h} - [(\mu + p)u_j u_h + pg_{jh}]_{;i}$$
$$+ \frac{5}{12}[g_{jh}(-\mu + 3p)_{;i} - g_{ji}(-\mu + 3p)_{;h}] \tag{3.33}$$

Now, if the fluid is shear-free, rotation-free, acceleration-free and the energy density and pressure are constant over the spacelike hyper-surface orthogonal to the fluid flow vector then using (3.20a), (3.22), (3.25), (3.28), (3.29) and (3.32) in (3.33) it can easily be verified that the divergence of concircular curvature tensor vanishes.

It may be noted that vanishing of shear, rotation and acceleration of the fluid and constantness of energy density and pressure over the spacelike hyper-surface orthogonal to the fluid flow vector are the conditions for a spacetime to represent Friedmann-Robertson-Walker cosmological model provided that $(\mu + p) \neq 0$.

Summing up these discussions, we can therefore state the following:

Theorem 10 *The necessary and sufficient condition for a perfect fluid spacetime to have divergence-free concircular curvature tensor is that either* $(\mu + p) = 0$ (*which is an Einstein space*) *or the spacetime is Friedmann-Robertson-Walker model satisfying* $(\mu - 3p) =$ *constant*.

We conclude this section with following:

Theorem 11 *For a spacetime with purely electromagnetic distribution, the divergence of concircular curvature tensor vanishes if and only if the energy-momentum tensor is of Codazzi type.*

The proof follows from (2.20) and (3.7).

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